Measuring net effects in signed ecological and social networks

Carlos Gómez-Ambrosi[1, 2] and Violeta Calleja-Solanas[3]

[1]*Department of Mathematics, University of Zaragoza, 50009 Zaragoza, Spain*
[2]*GOTHAM lab, Institute of Biocomputation and Physics of Complex*
Systems (BIFI), University of Zaragoza, 50018 Zaragoza, Spain
[3]*Department of Ecology and Evolution, Doñana Biological Station (EBD-CSIC), 41092 Seville, Spain*[*]

With improvements in data resolution and quality, researchers can now construct detailed representations of complex systems as signed, weighted, and directed networks. In this article, we introduce a framework for measuring net and indirect effects without simplifying these information-rich networks. Building on a generalization of Katz centrality, this framework captures both direct and indirect interactions, the effect of the whole network on a node and its reverse, the effect of a node on the whole network, while accommodating the complexity of signed, weighted, and directed edges.

To contextualize our contribution, we propose a taxonomy that unifies existing approaches and measures from the literature. We then apply our measure to ecological networks, where net and indirect effects remain critical yet difficult to quantify factors influencing coexistence. Specifically, we observe a strong correlation between negative net effects and species extinction in generalized Lotka-Volterra dynamics. Additionally, we test our framework on a real-world social network, where it effectively identifies informative importance rankings, providing insights into influence propagation and power dynamics.

I. INTRODUCTION

Complex networks provide a powerful framework for encoding pairwise interactions between entities, represented as nodes and links. For instance, in ecological networks, nodes represent species within a community, and a link a_{ij} denotes the direct effect of species j on the per-capita growth rate of species i. Similarly, in social networks, nodes often represent agents, while weighted links characterize their relationships' strength and nature. The structure of these networks is important to characterize because it affects their functioning. In ecology, interaction matrices underpin fundamental processes such as species coexistence [1] and biodiversity maintenance [2]. In social systems, network topology affects contagion processes [3] and the dynamics of influence and power [4], among other topics.

A common strategy to characterize network structure involves computing centrality measures that assess the importance of nodes within the network. Examples include closeness centrality, which captures how efficiently a node can access others, and betweenness centrality, which measures the extent to which a node acts as a bridge between other nodes [5]. Other centralities, such as Katz centrality [6] and PageRank [7], incorporate the influence of network structure to assess the importance of nodes. However, their application is often limited to unsigned and unweighted networks, constraining their relevance for many real-world systems. Signed networks, where links can represent positive or negative interactions (e.g., mutualism or competition in ecological networks, or trustfulness in social networks), require a more subtle interpretation. Weighted networks, which assign varying strengths to interactions, add an additional layer of complexity. Similarly, the directionality of links —essential for understanding cause-and-effect relationships— is typically underexplored in combination with sign and weight. While existing studies (e.g., [8, 9] have attempted to extend these measures to incorporate such features, significant gaps remain, particularly in their ability to provide a complete picture of node influence and network dynamics.

An equally important limitation of traditional centrality measures lies in their inability to separate indirect from direct effects. In many complex networks, the influence of a node extends beyond its immediate connections, propagating through paths of varying length and significance. For instance, ignoring indirect ecological interactions can lead to oversights in conservation strategies or misinterpretations of ecosystem stability [10, 11]. In social systems, indirect

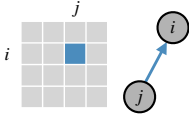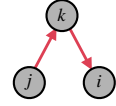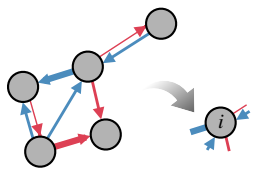---

[*] violeta.calleja@ebd.csic.es

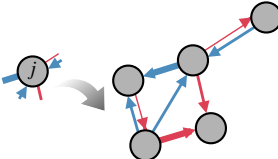|	Layout	Formula	Definition
Direct effects		$A = [a_{ij}]$	a_{ij} measures the effect that agent j has on agent i without the intervention of any intermediaries
Indirect effects		$a_{ik}a_{kj}$	Direct effects are multiplied along pathways to yield indirect effects («the enemy of my enemy is my friend»)
Net effects (Katz matrix)		$\mathcal{K} = \mathcal{K}(A, \alpha)$	Indirect effects are added among all possible pathways (together with direct effects) to yield net effects
In-Katz net effects		$[\mathcal{K}_{in}]_i = \sum_j \mathcal{K}_{ij}$	Net effect of the whole network on agent i
Out-Katz net effects		$[\mathcal{K}_{out}]_j = \sum_i \mathcal{K}_{ij}$	Net effect of agent j on the whole network

FIG. 1. Overview of the fundamental notions developed in Section II.

effects are central to phenomena like the spread of influence, where a person's actions may have ripple effects through their extended network [12]. Current centrality measures, such as Katz's and PageRank, account for indirect effects only in the context of the cumulative effect of the network on a single node and cannot disentangle their contribution to the final measure. By doing so, we risk overlooking the full extent of node influence.

To address these limitations, we propose shifting the focus from traditional centrality to a more comprehensive notion of net effect. This perspective allows us to capture the interconnected nature of networks and its consequences on their dynamics. Specifically, we introduce a framework to compute a net effect matrix that enables us to (1) measure the effect of the network on a node (as in classical centralities), (2) quantify the reverse measure, i.e. the effect of a node on the network, and (3) disentangle direct effects from indirect effects (Figure 1). Our approach builds on generalizing the Katz measure to account for signed, weighted, and directed networks. We bridge gaps in existing methodologies and provide tools to apply to a broad range of networks and questions. Finally, we demonstrate the utility of our framework through illustrative applications to ecological and social networks.

This paper is divided into three main sections. In Section II, we present a taxonomy and review of measures, and provide the mathematical details; in Section III we apply the measures to ecological networks; and in Section IV we apply them to social networks.

II. MEASURES ON SIGNED WEIGHTED DIRECTED NETWORKS

In this section, we extend classical Katz centrality to signed weighted directed networks and relate it to other centrality measures like Hubbell centrality and PageRank. See Figure 1 for an overview of the fundamental notions developed in this section.

A. Net effects

Our starting point is a signed weighted directed network with n nodes, as represented by its (signed weighted) adjacency matrix $A = [a_{ij}]$. Entry a_{ij} of A can take any real value, and represents the direct effect of node j on node i. We assume that our network does not contain self-loops, i.e. that $a_{ii} = 0$ for all i. Our intended interpretation is that the nodes in the network interact, and that a_{ij} measures the strength of the interaction $i \leftarrow j$ (including whether this interaction is positive or negative). Since A represents direct effects or interaction strengths rather than mere adjacency, we call it the matrix of direct effects or the interaction matrix of the network.

If $i \leftarrow k \leftarrow j$ is a (directed) walk of length 2 from node j to node i, with node k as its middle node, then the indirect effect of node j on node i along this walk is defined as $a_{ik}a_{kj}$, i.e. direct effects are multiplicative along walks. This complies with the rule that "the enemy of my enemy is my friend", in accordance with the usual interpretation in ecological as well as in social networks. In general, if $i \leftarrow k_{\ell-1} \leftarrow \cdots \leftarrow k_2 \leftarrow k_1 \leftarrow j$ is a walk of length $\ell > 1$ from node j to node i, then the indirect effect of node j on node i along this walk is defined as $a_{ik_{\ell-1}} \cdots a_{k_2 k_1} a_{k_1 j}$.

Continuing in this way, we can define the indirect effect of node j on node i of order $\ell > 1$, as the sum of the indirect effects of node j on node i along all walks of length ℓ from node j to node i, so that indirect effects are additive among walks. Notice that this is precisely the (i, j)-entry of the ℓ-th power of A:

$$[A^\ell]_{ij} = \sum_{k_1, \ldots, k_{\ell-1}=1}^{n} a_{ik_{\ell-1}} \cdots a_{k_2 k_1} a_{k_1 j}. \tag{1}$$

Hence A^ℓ is the matrix of indirect effects of order $\ell > 1$, and direct effects can be considered as indirect effects of order 1.

Finally, it seems natural to define the net effect of node j on node i as the sum of the indirect effects of node j on node i of all orders, i.e.

$$\sum_{\ell=1}^{\infty} [A^\ell]_{ij}. \tag{2}$$

The convergence of this number series for all i and j is equivalent to the convergence of the matrix series

$$\sum_{\ell=1}^{\infty} A^\ell, \tag{3}$$

in which case we have the equality

$$\sum_{\ell=1}^{\infty} [A^\ell]_{ij} = [\sum_{\ell=1}^{\infty} A^\ell]_{ij}. \tag{4}$$

We are thus interested in characterizing the convergence of the matrix series in Eq. 3.

In the sequel, I_n denotes the $n \times n$ identity matrix and $\rho(\cdot)$ denotes the spectral radius of a matrix. By a well-known result in matrix analysis (see e.g. [13], p.618), the Neumann series $I_n + A + A^2 + \cdots$ converges if and only if $\rho(A) < 1$, in which case $(I_n - A)^{-1}$ exists and has a Neumann series expansion

$$(I_n - A)^{-1} = \sum_{\ell=0}^{\infty} A^\ell. \tag{5}$$

Notice that this expansion adds an extra term I_n to the matrix series in Eq. 3.

Thus, net effects are well defined under the hypothesis that $\rho(A) < 1$, but not otherwise. What classical measures

like Katz's and PageRank do is rescale the matrix of direct effects so that the above definition of net effects makes sense for the rescaled matrix. There are three natural ways of achieving this for signed weighted directed networks, which will be addressed in Sections II B, II C, and II D.

B. Katz measure

Let α be a positive real number, and let us rescale all the direct effects by α, obtaining a rescaled matrix of direct effects $\alpha A = [\alpha a_{ij}]$. We call α a global rescaling parameter, since it rescales all the direct effects of the network.

If we choose $\alpha < 1/\rho(A)$, then $\rho(\alpha A) = \alpha \rho(A) < 1$, and hence $(I_n - \alpha A)^{-1}$ exists and has a Neumann series expansion. This allows us to define the Katz matrix of net effects of A with respect to α as

$$\mathscr{K}(A, \alpha) = -I_n + (I_n - \alpha A)^{-1} = \sum_{\ell=1}^{\infty} \alpha^\ell A^\ell. \tag{6}$$

In the light of Section II A,

$$[\mathscr{K}(A, \alpha)]_{ij} = \sum_{\ell=1}^{\infty} \alpha^\ell [A^\ell]_{ij} \tag{7}$$

measures the net effect of node j on node i, taking into account that the indirect effect of node j on node i along a walk of length ℓ has been "attenuated" by a factor of α^ℓ.

Starting from the Katz matrix, we can construct two vectors: the in-Katz vector of net effects

$$\mathscr{K}_{\mathrm{in}}(A, \alpha) = (-I_n + (I_n - \alpha A)^{-1})\, \mathbf{1}, \tag{8}$$

and the out-Katz vector of net effects

$$\mathscr{K}_{\mathrm{out}}(A, \alpha) = \mathbf{1}^T(-I_n + (I_n - \alpha A)^{-1}), \tag{9}$$

where $\mathbf{1}$ represents the (column) vector of all ones and a T superscript denotes transposition. These vectors are, respectively, the vector of row sums and the vector of column sums of the Katz matrix. Taking components, we can define the in-Katz net effect on node i as

$$[\mathscr{K}_{\mathrm{in}}(A, \alpha)]_i = \sum_{j=1}^{n} [\mathscr{K}(A, \alpha)]_{ij}, \tag{10}$$

which measures the net effect of the whole network on node i. In case the network is unsigned and unweighted, this is the classical Katz centrality [6] of node i. Similarly, we can define the out-Katz net effect of node j as

$$[\mathscr{K}_{\mathrm{out}}(A, \alpha)]_j = \sum_{i=1}^{n} [\mathscr{K}(A, \alpha)]_{ij}, \tag{11}$$

which measures the net effect of node j on the whole network. Notice that $\mathscr{K}_{\mathrm{out}}(A, \alpha) = [\mathscr{K}_{\mathrm{in}}(A^T, \alpha)]^T$, since A^T represents the reversed of the original network (i.e. the direction of every link is reversed, but preserving its weight and sign).

Hubbell measures are a slight variation of Katz's (see [14], and can be defined by the formulas below under the

same hypothesis that $\alpha < 1/\rho(A)$:

$$\mathcal{H}(A, \alpha) = (I_n - \alpha A)^{-1} = \sum_{\ell=0}^{\infty} \alpha^\ell A^\ell = \mathcal{K}(A, \alpha) + I_n, \tag{12}$$

$$\mathcal{H}_{\text{in}}(A, \alpha) = (I_n - \alpha A)^{-1} \mathbf{1} = \mathcal{K}_{\text{in}}(A, \alpha) + \mathbf{1}, \tag{13}$$

and

$$\mathcal{H}_{\text{out}}(A, \alpha) = \mathbf{1}^T (I_n - \alpha A)^{-1} = \mathcal{K}_{\text{out}}(A, \alpha) + \mathbf{1}^T. \tag{14}$$

In particular, $[\mathcal{H}_{\text{in}}(A, \alpha)]_i$ is the classical Hubbell centrality [15] of node i in case the "boundary condition" vector is $\mathbf{1}$.

There is a trick to choosing $\alpha < 1/\rho(A)$ without having to compute $\rho(A)$ in advance. Let $C_j(B)$ denote the sum of the absolute values of the non-diagonal entries in the j-th column of a matrix B, and assume that $I_n - \alpha A$ is strictly diagonally dominant (SDD) by columns, i.e.

$$C_j = C_j(I_n - \alpha A) = C_j(\alpha A) = \sum_{i \neq j} |\alpha a_{ij}| = \alpha \sum_{i \neq j} |a_{ij}| < 1. \tag{15}$$

This can be achieved by choosing $\alpha < 1/(n-1)M$, where

$$M \geq \max_{i,j} |a_{ij}|. \tag{16}$$

By Gershgorin circle theorem (see e.g. [13], p.498), every eigenvalue of αA lies within at least one of the Gershgorin discs $\overline{D}(0, C_j)$, hence $\rho(\alpha A) < 1$.

C. PageRank measure

Now, let α and $d_1, \ldots, d_n$ be positive real numbers, and let us rescale the direct effect of node j on node i by αd_j^{-1} for all i and j, obtaining a rescaled matrix of direct effects $\alpha A D^{-1} = [\alpha a_{ij} d_j^{-1}]$, where $D = \text{diag}(d_1, \ldots, d_n)$. We call d_j a local rescaling parameter, since d_j^{-1} rescales the direct effect that node j has on its neighboring nodes. Equivalently, d_j^{-1} rescales column j of the interaction matrix. As in Section II B, α is a global rescaling parameter.

If we choose α and $d_1, \ldots, d_n$ so that $\rho(\alpha A D^{-1}) < 1$ (we explain below how to do this), then $(I_n - \alpha A D^{-1})^{-1}$ exists and has a Neumann series expansion, allowing us to mimic the definitions of the previous section. More precisely, the PageRank matrix of net effects of matrix A with respect to the rescaling parameters α and $d_1, \ldots, d_n$, and the corresponding in- and out-PageRank vectors, can be defined, respectively, by

$$\text{PR}(A, \alpha, d_1, \ldots, d_n) = (I_n - \alpha A D^{-1})^{-1} = \sum_{\ell=0}^{\infty} \alpha^\ell (A D^{-1})^\ell, \tag{17}$$

$$\text{PR}_{\text{in}}(A, \alpha, d_1, \ldots, d_n) = (I_n - \alpha A D^{-1})^{-1} \mathbf{1}, \tag{18}$$

and

$$\text{PR}_{\text{out}}(A, \alpha, d_1, \ldots, d_n) = \mathbf{1}^T (I_n - \alpha A D^{-1})^{-1}. \tag{19}$$

Notice that the PageRank measures of network A are simply the Hubbell measures of the locally rescaled network AD^{-1}.

There are essentially two ways of choosing the parameters so that $\rho(\alpha AD^{-1}) < 1$. The first one is to take $\alpha < 1$ and choose each d_j so that

$$d_j \geq \sum_{i \neq j} |a_{ij}| = d_j^{\text{out}}, \tag{20}$$

the weighted out-degree of node j. Then $I_n - \alpha AD^{-1}$ is SDD by columns, i.e.

$$C_j(I_n - \alpha AD^{-1}) = C_j(\alpha AD^{-1}) = \alpha\, d_j^{-1} \sum_{i \neq j} |a_{ij}| \leq \alpha < 1, \tag{21}$$

and Gershgorin theorem applies as in Section II B. If the network is unsigned and unweighted, and d_j^{out} is positive for all j (i.e. there are no "dangling nodes"), then

$$\frac{1-\alpha}{n}\, \text{PR}_{\text{in}}(A, \alpha, d_1^{\text{out}}, \ldots, d_n^{\text{out}}) \tag{22}$$

is classical PageRank centrality with "teleportation parameter" α and uniform "teleportation distribution vector" $\frac{1}{n}\mathbf{1}$ (see [7] for details). On the other hand, if the network is unsigned (but possibly weighted), and all d_j^{out} are positive, then

$$\frac{1-\alpha}{n}\, \text{PR}_{\text{out}}(A, \alpha, d_1^{\text{out}}, \ldots, d_n^{\text{out}}) = \frac{1}{n}\mathbf{1}^T, \tag{23}$$

so that the out-PageRank vector does not contain useful information in this case.

The second way is to take $\alpha < 1/(n-1)$ and choose each d_j so that $d_j \geq |a_{ij}|$ for all i. Then

$$C_j(\alpha AD^{-1}) = \alpha\, d_j^{-1} \sum_{i \neq j} |a_{ij}| \leq \alpha(n-1) < 1, \tag{24}$$

and Gershgorin theorem applies again.

D. Reverse PageRank measure

Finally, let α and $d_1, \ldots, d_n$ be positive real numbers, and let us rescale the direct effect of node j on node i by αd_i^{-1} for all i and j, obtaining a rescaled matrix of direct effects $\alpha D^{-1}A = [\alpha d_i^{-1} a_{ij}]$, where $D = \text{diag}(d_1, \ldots, d_n)$. As in Section II C, we call d_i a local rescaling parameter, since d_i^{-1} rescales the direct effect that its neighboring nodes have on node i. Equivalently, d_i^{-1} rescales row i of the interaction matrix.

If we choose α and $d_1, \ldots, d_n$ so that $\rho(\alpha D^{-1}A) < 1$, then the reverse PageRank matrix of net effects of matrix A with respect to the rescaling parameters α and $d_1, \ldots, d_n$, and the corresponding in- and out-reverse PageRank vectors, can be defined, respectively, by

$$\text{RPR}(A, \alpha, d_1, \ldots, d_n) = (I_n - \alpha D^{-1}A)^{-1} = \sum_{\ell=0}^{\infty} \alpha^\ell (D^{-1}A)^\ell, \tag{25}$$

$$\text{RPR}_{\text{in}}(A, \alpha, d_1, \ldots, d_n) = (I_n - \alpha D^{-1}A)^{-1}\mathbf{1}, \tag{26}$$

and

$$\text{RPR}_{\text{out}}(A, \alpha, d_1, \ldots, d_n) = \mathbf{1}^T (I_n - \alpha D^{-1} A)^{-1}. \tag{27}$$

Similarly to Section II C, the reverse PageRank measures of network A are simply the Hubbell measures of the locally rescaled network $D^{-1}A$.

Reverse PageRank measures can be derived directly from PageRank measures of the reversed network (hence their name):

$$\text{RPR}(A, \alpha, d_1, \ldots, d_n) = \text{PR}(A^T, \alpha, d_1, \ldots, d_n)^T, \tag{28}$$

$$\text{RPR}_{\text{in}}(A, \alpha, d_1, \ldots, d_n) = \text{PR}_{\text{out}}(A^T, \alpha, d_1, \ldots, d_n)^T, \tag{29}$$

and

$$\text{RPR}_{\text{out}}(A, \alpha, d_1, \ldots, d_n) = \text{PR}_{\text{in}}(A^T, \alpha, d_1, \ldots, d_n)^T. \tag{30}$$

As in the case of PageRank, there are essentially two ways of choosing the parameters so that $\rho(\alpha D^{-1} A) < 1$. The first one is to take $\alpha < 1$ and choose each d_i so that

$$d_i \geq \sum_{j \neq i} |a_{ij}| = d_i^{\text{in}}, \tag{31}$$

the weighted in-degree of node i. If $R_i(B)$ denotes the sum of the absolute values of the non-diagonal entries in the i-th row of a matrix B, then $I_n - \alpha D^{-1} A$ is SDD by rows, i.e.

$$R_i = R_i(I_n - \alpha D^{-1} A) = R_i(\alpha D^{-1} A) = \alpha \, d_i^{-1} \sum_{j \neq i} |a_{ij}| \leq \alpha < 1. \tag{32}$$

Since Gershgorin theorem is symmetric with respect to rows and columns, every eigenvalue of $\alpha D^{-1} A$ lies within at least one of the Gershgorin discs $\overline{D}(0, R_i)$, and hence $\rho(\alpha D^{-1} A) < 1$. If the network is unsigned and unweighted, and $d_i^{\text{in}}(A) = d_i^{\text{out}}(A^T)$ is positive for all i, then

$$\text{RPR}_{\text{out}}(A, \alpha, d_1^{\text{in}}(A), \ldots, d_n^{\text{in}}(A)) = \text{PR}_{\text{in}}(A^T, \alpha, d_1^{\text{out}}(A^T), \ldots, d_n^{\text{out}}(A^T))^T \tag{33}$$

is the classical definition of reverse PageRank as the PageRank measure of the reversed network (see [7] for details).

The second way is to take $\alpha < 1/(n-1)$ and choose each d_i so that $d_i \geq |a_{ij}|$ for all j. Then

$$R_i(\alpha D^{-1} A) = \alpha \, d_i^{-1} \sum_{j \neq i} |a_{ij}| \leq \alpha(n-1) < 1, \tag{34}$$

and the row version of Gershgorin theorem applies as before.

Since all the measures introduced so far are defined in terms of matrix series, they can be approximated by their truncated versions. For example, the L-truncated out-reverse PageRank vector would be defined by

$$\text{RPR}_{\text{out}}^{\text{L}}(A, \alpha, d_1, \ldots, d_n) = \mathbf{1}^T \sum_{\ell=0}^{\text{L}} \alpha^\ell (D^{-1} A)^\ell. \tag{35}$$

This definition will be used in Section IV B.

III.   APPLICATIONS TO ECOLOGICAL NETWORKS

In this section, we consider an ecological network of n species whose dynamics are governed by generalized Lotka-Volterra (GLV) equations [16] of the form

$$\dot{x}_i = x_i \left(r_i + \sum_{j=1}^{n} a_{ij} x_j \right), \quad i = 1, \ldots, n, \tag{36}$$

where x_i is the abundance of species i, $\dot{x}_i$ denotes its time derivative, which is the growth rate of species i, r_i is the intrinsic growth rate of species i, and a_{ij} measures the direct effect of species j on the per capita growth rate of species i, i.e. on $\dot{x}_i/x_i$. The intra-specific coefficient a_{ii} accounts for the self-regulation of species i, whereas the inter-specific coefficients a_{ij}, for $j \neq i$, account for the pairwise interactions between species. Depending on the signs of a_{ij} and a_{ji}, these interactions can be predator-prey $(+ -)$, competitive $(- -)$, mutualistic $(+ +)$, etc. [17]. The matrix $A = (a_{ij})$ is the interaction matrix of the GLV equations [18], but it cannot be considered as the interaction matrix of our ecological network in the sense of Section II, since in general it is a matrix with a non-zero diagonal.

Three simplifying hypotheses will be applied to the setting of the preceding paragraph: (i) all species have negative self-regulation, i.e. $a_{ii} < 0$ for all i; (ii) the direct effect that species j has on itself (in absolute value) is greater than the direct effect that species j has on species i (in absolute value), i.e. $-a_{jj} > |a_{ij}|$ for all i and j, $i \neq j$; and (iii) all r_i are equal, say $r_i = r$ for all i. Hypothesis (ii) is sometimes replaced by the stronger assumption that A is SDD by columns, and Hypothesis (iii) is assumed to focus on the effect of the interactions over the dynamics. See e.g. [19] for a discussion of these hypotheses.

Given the interaction matrix $A = (a_{ij})$ of the GLV equations, let us denote by $A^\dagger = (a_{ij}^\dagger)$ the matrix given by

$$a_{ij}^\dagger = a_{ij} \quad \text{if } i \neq j \quad \text{and} \quad a_{ii}^\dagger = 0 \quad \text{for all } i. \tag{37}$$

In general, we will use a dagger to denote the operation of resetting all the diagonal entries of a matrix to zero. Let $D = \text{diag}(-a_{11}, \ldots, -a_{nn})$, so that $A = -D + A^\dagger$, and let $B = AD^{-1} = -I_n + A^\dagger D^{-1} = -I_n + B^\dagger$. Notice that the diagonal entries in D are positive by Hypothesis (i), hence D is invertible, and that $B = (b_{ij}) = (-a_{ij}/a_{jj})$. Matrix $B^\dagger$ will be considered as the interaction matrix of our ecological network. Matrix $A^\dagger$ is not suitable, since it lacks the information contained in the self-regulation coefficients. Notice that entry $b_{ij}^\dagger$ of $B^\dagger$ measures the direct effect of species j on species i relative to the direct effect (in absolute value) that species j has on itself.

Matrix B is the beta-matrix of [20] (see also [18]. The choice of $B^\dagger$ over $A^\dagger$ as the matrix of direct effects of our ecological network also has the advantage that, whereas $a_{ij}^\dagger$ has dimension of [population^{-1}time^{-1}], $b_{ij}^\dagger$ is non-dimensional. This allows us to properly combine indirect effects of different orders into a single net effect, by transforming interactions into relative interactions. See [21] for a detailed discussion of this issue.

Next, we change the variables in Eqs. 36 to $y_i = -a_{ii}x_i$. Since all a_{ii} are negative by Hypothesis (i), this can be interpreted as a rescaling of the way in which we measure the abundance of each species separately. The resulting GLV equations are

$$\dot{y}_i = y_i \left(r_i + \sum_{j=1}^{n} \left(-\frac{a_{ij}}{a_{jj}} \right) y_j \right), \quad i = 1, \ldots, n, \tag{38}$$

in which the interaction matrix (of the new GLV equations) is precisely B. It is apparent that the dynamics of Eqs. 36 and 38 are similar, since we have just rescaled the variables by positive constants.

Finally, we change the variables in Eqs. 38 to $z_i = y_i(y_1 + \cdots + y_n)^{-1}$, i.e. to relative abundances. Having into

account Hypothesis (iii) and after a change in velocity, we obtain that the z_i satisfy the replicator equations [16]

$$\dot{z}_i = z_i((B\mathbf{z})_i - \mathbf{z}^T B\mathbf{z}), \quad i = 1, \ldots, n, \tag{39}$$

on the n-dimensional simplex S_n, where $\mathbf{z} = (z_1, \ldots, z_n)^T$ and $(B\mathbf{z})_i$ denotes the i-th component of vector $B\mathbf{z}$. This is Exercise 7.5.2 of [16]. Eqs. 39 have the advantage over Eqs. 38 of confining the dynamics to a bounded phase space, and are equally effective for detecting the extinction of species.

A. Katz measure as a structural predictor of species extinction

Following the introduction to Section III, $B = -I_n + B^\dagger$ is the interaction matrix of the GLV and replicator equations, whereas $B^\dagger$ is the interaction matrix of our ecological network. As a structural predictor of species extinction, we propose the in-Katz vector of net effects $\mathcal{K}_{\text{in}}(B^\dagger, \alpha)$ from Section II B. Notice that, by Hypothesis (ii), $|b_{ij}^\dagger| < 1$ for all i and j, hence we can take $M = 1$ and $\alpha < 1/(n-1)$ in the definition of the in-Katz vector. We have opted for $\alpha = 1/n$ as a sensible choice. Notice in passing that, since $B^\dagger = A^\dagger D^{-1}$,

$$\mathcal{K}_{\text{in}}(B^\dagger, \alpha) = \text{PR}_{\text{in}}(A^\dagger, \alpha, -a_{11}, \ldots, -a_{nn}) - \mathbf{1}, \tag{40}$$

and that, again by Hypothesis (ii), a choice of $\alpha < 1/(n-1)$ fits into the definition of the in-PageRank vector above (see Section II C).

We claim that the species that suffer a negative net effect from the network (and then negative values of in-Katz net effect) have a higher probability of becoming extinct under the GLV dynamics or, equivalently, under the replicator dynamics.

In order to test our claim, we have averaged over 1000 realizations of the following process for different communities' sizes and network connectances. We begin by selecting a connected matrix $B = -I_n + B^\dagger$ with off-diagonal entries randomly chosen from a uniform distribution in the interval $(-1, 1)$, once a proportion c of pairs (i, j) and (j, i), with $j \neq i$, have been selected for non-zero interaction. Then we compute $\mathcal{K}_{\text{in}}(B^\dagger, 1/n)$, and store the species with minimum (negative) in-Katz net effect, as well as those with negative in-Katz net effect. Next, we sample an initial condition uniformly from the simplex S_n, and numerically integrate the replicator Eq. 39 using the Runge–Kutta method of order 4 until the system stabilizes. At each time step, we check for species whose densities have fallen below $10^{-5}/n$, which is the threshold density we have chosen for a species to be considered extinct, and reset their densities to zero, as well as the corresponding rows and columns of matrix B. For the sake of numerical stability, we normalize the resulting vector of densities to replace it within the simplex. At the end of the integration procedure, we count the number of extinct species, we check whether the species with minimum in-Katz net effect has become extinct, and we calculate the proportion of species with negative in-Katz net effect that have become extinct. The results are displayed in Figure 2.

B. A couple of "evolutionary principles"

A global numerical measure that we can consider for a signed, weighted directed network A is the mean Katz net effect (with respect to α), defined as

$$\langle \mathcal{K}(A, \alpha) \rangle = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\mathcal{K}(A, \alpha)]_{ij}. \tag{41}$$
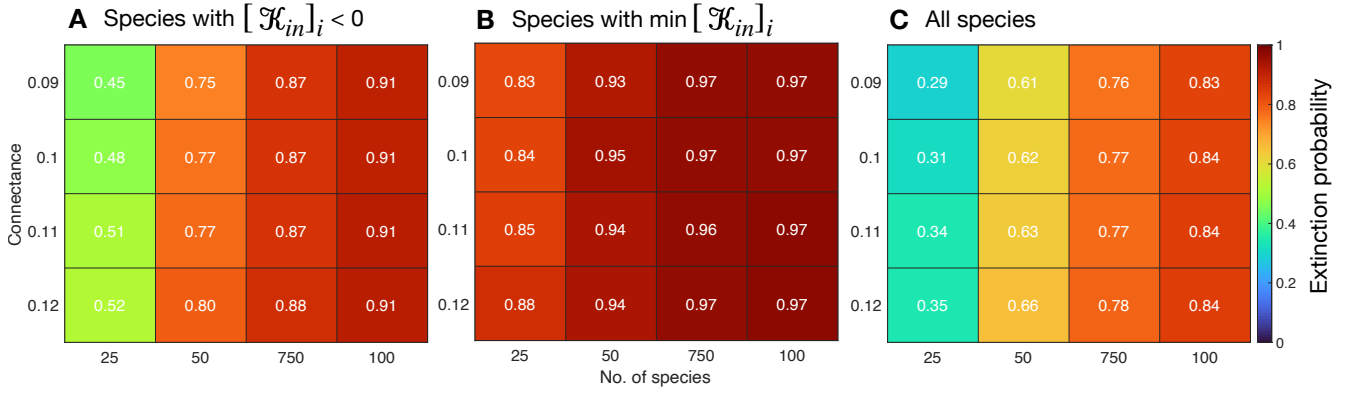
FIG. 2. Average extinction probability for (A) a species with negative in-Katz net effect, (B) the species with minimum in-Katz net effect, and (C) all species.
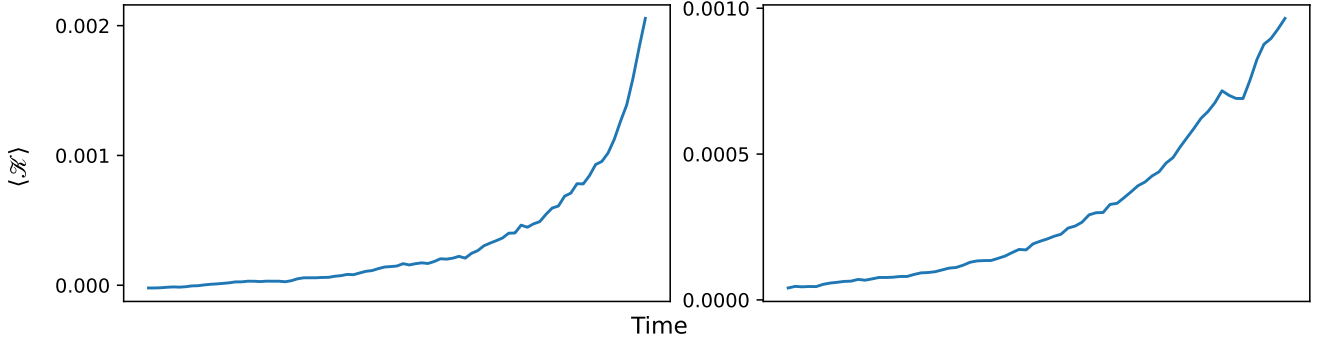

FIG. 3. Two typical examples of how the mean Katz net effect increases on average as species become extinct under a GLV dynamics.

If $B = -I_n + B^\dagger$ is one of the beta-matrices selected in the process described at the end of the previous section, and we compute $\langle \mathfrak{K}(B^\dagger, 1/n) \rangle$, the result is roughly zero. But if we recalculate the mean Katz net effect along the process for the matrix that results every time a species becomes extinct and the corresponding row and column is deleted, we obtain a function that is increasing on average. Typical examples are shown in Figure 3. This suggests that an "evolutionary optimization principle" might be at work in the evolution of species under GLV dynamics, in the sense that the mean Katz net effect is optimized on average along the process.

Related to the above, we have also noticed the following fact. Given the manner in which matrix B has been selected, on average one-half of the interactions in the ecological network represented by $B^\dagger$ are predator–prey, one-quarter are competitive, and one-quarter are mutualistic. But even if we change these proportions in our simulations, we observe that, as more species become extinct, competitive interactions tend to disappear almost altogether, and mutualistic interactions tend to preponderate. This suggests that a "generalized competitive exclusion principle" might be also at work in the evolution of species under GLV dynamics.

C.   Relation with the net effects matrix of press perturbations

Let us assume that $A = (a_{ij})$, the interaction matrix of our ecological system as in the introduction to Section III, satisfies the strong version of Hypothesis (ii), i.e. it is SDD by columns. It is well-known that A is then an invertible matrix (this is an immediate consequence of Gershgorin theorem, and it is usually known as the Lévy-Desplanques theorem). The matrix $-A^{-1}$ is known in ecology as the net effects matrix of the interaction matrix A, and it is

important in the study of press perturbations (see [18] for details). Assuming Hypothesis (i), let us change our perspective to the matrix $B = (b_{ij}) = (-a_{ij}/a_{jj}) = -I_n + B^\dagger$ of relative direct effects. Since A is SDD by columns, i.e.

$$-a_{jj} > \sum_{i \neq j} |a_{ij}|, \tag{42}$$

we have that

$$C_j(B^\dagger) = \sum_{i \neq j} |b_{ij}| = -a_{jj}^{-1} \sum_{i \neq j} |a_{ij}| < 1, \tag{43}$$

so that $\rho(B^\dagger) < 1$ as at the end of Section II B, and hence $\mathscr{H}(B^\dagger, 1)$ is well-defined. Finally,

$$-B^{-1} = -(-I_n + B^\dagger)^{-1} = (I_n - B^\dagger)^{-1} = \mathscr{H}(B^\dagger, 1). \tag{44}$$

This reinforces the importance of Vandermeer's beta-matrix as a relevant tool in the study of ecological networks.

IV. APPLICATIONS TO SOCIAL NETWORKS

In this section, we review some of the previous attempts at defining a measure of centrality or of net effects for signed networks that have appeared in social network analysis.

The first attempt is the status score of [22], based on classical eigenvector centrality. As noted in [14], this definition is problematic, as a matrix with positive and negative entries does not need to have a dominant eigenvalue. Like in the subsequent papers, these works illustrate their analysis with data from Sampson's well-known study of relationships of monks in a monastery [23], a key example for signed network analysis, much like Zachary's karate club network for community detection [24]. We remark, for future reference, that there were four monks —Basil, Elias, Simplicius, and Gregory— who were expelled from the monastery.

A. PN centrality

The second attempt is the PN centrality of [14]. They assume that their network is signed, but neither weighted nor directed, i.e. $A = [a_{ij}]$ is symmetric and $a_{ij} = 0, 1$ or -1 ($a_{ii} = 0$ for all i). Let $B = [b_{ij}]$ be defined by $b_{ij} = 1/2$ if $a_{ij} = 1$, and $b_{ij} = a_{ij}$ otherwise, i.e. we weigh every positive link by $1/2$, so that B is obtained from A by rescaling the positive entries. Let also $\alpha = 1/(n-1)$, and consider the matrix αB. Under the hypothesis that the positive degree of every node is positive, i.e. that every column of B contains at least one $1/2$ entry,

$$C_j(\alpha B) = \sum_{i \neq j} |\alpha b_{ij}| = \alpha \sum_{i \neq j} |b_{ij}| < \alpha \sum_{i \neq j} |a_{ij}| \leq \alpha(n-1) = 1, \tag{45}$$

so that $\rho(\alpha B) < 1$ by Gershgorin theorem. The PN centrality of A is then defined by

$$\text{PN}(A) = \mathscr{H}_{\text{in}}(B, 1/(n-1)). \tag{46}$$

We note that the interpretation of this definition is opaque since they rescale asymmetrically positive and negative links with no clearly stated justification. What is more, the good qualities that Everett and Borgatti attribute to $\text{PN}(A)$ in their analysis of the Sampson monastery are equally satisfied by $\mathscr{H}_{\text{in}}(A, 1/n)$. Their paper also contains in- and out- versions of PN centrality for the directed case, but we have not been able to frame their definitions in the

Monk ranking	NE		RPR$_{out}$ - 1	
18	Basil	-0.754	Basil	-1.097
17	Elias	-0.643	Elias	-0.904
16	Peter	-0.290	Simplicius	-0.476
15	Simplicius	-0.247	John	-0.210
14	John	-0.134	Peter	-0.077
13	Victor	-0.110	Amand	0.058
12	Amand	0.157	Victor	0.112
11	Berthold	0.156	Gregory	0.352
10	Albert	0.328	Berthold	0.383
9	Gregory	0.366	Albert	0.451
8	Louis	0.380	Louis	0.546
7	Romul	0.508	Mark	0.551
6	Mark	0.460	Romul	0.616
5	Hugh	0.576	Hugh	0.669
4	Boniface	0.727	Boniface	0.878
3	Ambrose	0.796	Winfrid	1.016
2	Winfrid	0.897	Bonaventure	1.090
1	Bonaventure	0.937	Ambrose	1.102

FIG. 4. Comparison of the monk's ranking according to the net effect truncated at walks of length 3 calculated in [9] —left column—, with the ranking when all walks are taken into account —right column—. The right column seems to better capture the actual ranking based on the finally expelled monks (in red).

picture presented in this paper.

B. Alternative definition of net effect

The third attempt that we would like to review is the alternative definition of net effect that appears in [9]. In the most general case, they assume that their network $A = [a_{ij}]$ is signed and directed (but not weighted), i.e. $a_{ij} = 0, 1$ or -1 ($a_{ii} = 0$ for all i), but A is not necessarily symmetric. After a careful translation of their calculations into matrix language (including a change from row notation $i \rightarrow j$ to column notation $i \leftarrow j$), their definition of the vector $\mathrm{NE}^{\mathrm{L}}(A)$ whose j-th component is the net effect that node j has on the whole network up to L steps with respect to the "weighting function" $g(\ell) = \alpha^\ell$ is

$$\mathrm{NE}^{\mathrm{L}}(A) = \mathrm{RPR}_{\mathrm{out}}^{\mathrm{L}}(A, \alpha, d_1^{\mathrm{in}}, \ldots, d_n^{\mathrm{in}}) - \mathbf{1}^T = \mathbf{1}^T \sum_{\ell=1}^{\mathrm{L}} \alpha^\ell (D^{-1}A)^\ell, \tag{47}$$

where d_i^{in} is the total in-degree of node i as in Eq. 31 and $D = \mathrm{diag}(d_1^{\mathrm{in}}, \ldots, d_n^{\mathrm{in}})$ (see Eq. 35 for the definition of $\mathrm{RPR}_{\mathrm{out}}^{\mathrm{L}}$). Since they consider g to be identically 1, we can set $\alpha = 1$ in Eq. 47.

In Figure 4, we display the net effect up to $L = 3$ steps calculated on the like-dislike relation network from Sampson monastery as it appears in [9], and compare it with the full version of $\mathrm{RPR}_{\mathrm{out}}(A, \alpha, d_1^{\mathrm{in}}, \ldots, d_n^{\mathrm{in}}) - \mathbf{1}^T$ for $\alpha = 0.9999 < 1$. The latter seems to reflect the monk's ranking more realistically in that the four monks who were expelled are worse positioned.

Monk ranking	$[\mathcal{K}_{in}]_i$	
18	Basil	-0.620
17	Simplicius	-0.594
16	Elias	-0.550
15	Amand	-0.326
14	Berthold	-0.059
13	Romul	-0.041
12	Albert	-0.030
11	Victor	-0.017
10	Peter	-0.005
9	Hugh	0.008
8	Louis	0.013
7	Mark	0.136
6	Boniface	0.173
5	Gregory	0.290
4	Winfrid	0.291
3	Ambrose	0.301
2	John	0.324
1	Bonaventure	0.527

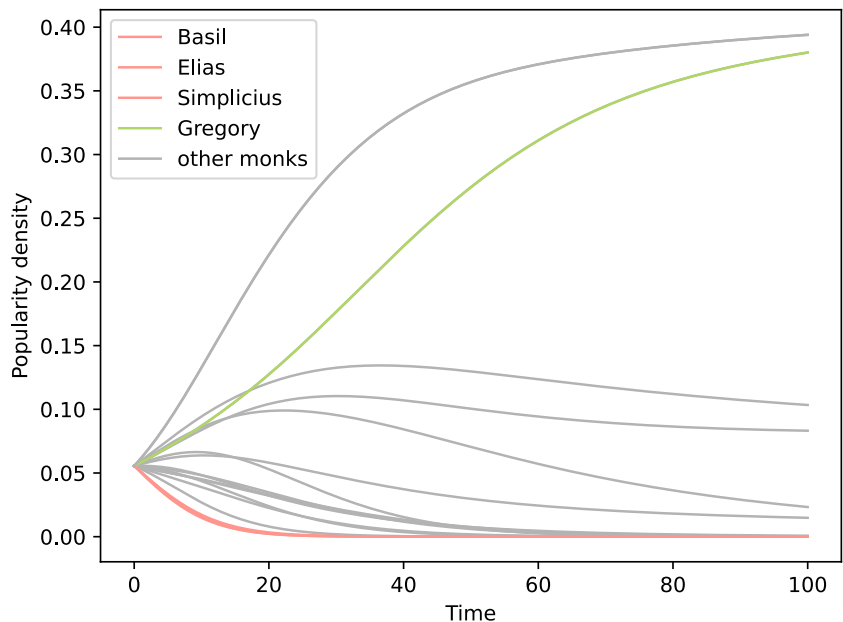FIG. 5. Raking of the monks given by the in-Katz measure of net effects and the evolution of their popularity as modeled by a replicator dynamics.

C. Social replicator dynamics

In this final section, we have applied the methods from Section III to the social network of the 18 monks of the Sampson monastery. We have taken (from the Ucinet software, see [8] the four matrices of positive interactions (LK3, ES, IN and PR) and the corresponding four matrices of negative interactions (DLK, DES, NIN and NPR), and we have formed the full matrix of interactions

$$A = \text{LK3}^T + \text{ES}^T + \text{IN}^T + \text{PR}^T - \text{DLK}^T - \text{DES}^T - \text{NIN}^T - \text{NPR}^T \tag{48}$$

(the matrices are transposed to comply with our column notation). Next, we have computed the in-Katz measure of net effects $\mathcal{K}_{in}(A, 1/18)$, whose values can be seen in the left panel of Figure 5. Of the four monks that were expelled from the monastery, this measure captures Basil, Elias and Simplicius as the three monks with the lowest (negative) in-Katz measure, but fails to capture Gregory, who has a (positive) high measure. Finally, identifying fitness with popularity and starting in the center of the simplex, we have run a replicator dynamics with interaction matrix A. In agreement with the in-Katz scores, Basil, Elias and Simplicius are the first three monks to "become extinct", while Gregory is one of the only four monks that "survive" in the end, as represented in the right panel of Figure 5.

V. DISCUSSION

Our generalization of Katz centrality provides a versatile tool for analyzing signed, weighted, and directed networks, addressing long-standing limitations in traditional centrality measures. We demonstrate its versatility to capture nuanced dynamics in complex networks, particularly by disentangling direct and indirect interactions. By quantifying

how agents affect and are affected by their networks, our approach offers a more complete perspective than traditional measures. In the ecological context, the correlation between negative in-Katz measures and species extinction, and its ability to identify species at risk due to indirect competitive pressures or destabilizing interactions provides a powerful lens for ecosystem conservation. Similarly, our results in social networks underscore the relevance of net effects for understanding phenomena such as influence propagation and power dynamics.

The inclusion of sign, weight, and direction in edges makes our framework broadly applicable to real-world networks where relationships are inherently heterogeneous. However, challenges remain. For example, our ecological applications have so far been tested on random interactions. Given the growing evidence that the non-random structure of interactions plays a critical role in species maintenance [25–27], studying real-world ecological interaction networks has become increasingly urgent. For instance, the role of indirect interactions in shaping coexistence and community assembly is especially intriguing [11, 28], and addressing this question requires a detailed analysis of signed and weighted networks. Not only that, integrating this framework with empirical datasets from other fields, such as economics or epidemiology, could reveal further insights into network behavior. By bridging gaps in methodology, our work opens roads for more comprehensive analyses of network structure and dynamics across disciplines.

CONFLICT OF INTEREST DISCLOSURE

The authors declare that they comply with the PCI rule of having no financial conflicts of interest in relation to the content of the article.

DATA, SCRIPT, CODE, AND SUPPLEMENTARY INFORMATION AVAILABILITY

Script and codes are available online at the Github repository.

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[1] P. Chesson, Mechanisms of maintenance of species diversity, Annual review of Ecology, Evolution, and Systematics **31**, 343 (2000).

[2] O. Godoy, I. Bartomeus, R. P. Rohr, and S. Saavedra, Towards the integration of niche and network theories, Trends in Ecology & Evolution **33**, 287 (2018).

[3] D. Soriano-Paños, L. Lotero, A. Arenas, and J. Gómez-Gardeñes, Spreading processes in multiplex metapopulations containing different mobility networks, Physical Review X **8**, 031039 (2018).

[4] A. Vendeville and F. Diaz-Diaz, Echo chamber effects in signed networks, arXiv preprint 10.48550/arXiv.2406.17435 (2024).

[5] M. Newman, *Networks*, 2nd ed. (Oxford University Press, 2018).

[6] L. Katz, A new status index derived from sociometric analysis, Psychometrika **18**, 39 (1953).

[7] D. F. Gleich, Pagerank beyond the web, SIAM Review **57**, 321 (2015).

[8] S. P. Borgatti, M. G. Everett, and L. C. Freeman, *Ucinet for Windows: Software for Social Network Analysis* (Harvard, MA: Analytic Technologies, 2002).

[9] W.-C. Liu, L.-C. Huang, C. W.-J. Liu, and F. Jordán, A simple approach for quantifying node centrality in signed and directed social networks, Applied Network Science **5**, 1 (2020).

[10] J. Wootton, The nature and consequences of indirect effects in ecological communities, Annual Review of Ecology and Systematics **25**, 443 (1994).

[11] P. R. Guimaraes Jr, M. M. Pires, P. Jordano, J. Bascompte, and J. N. Thompson, Indirect effects drive coevolution in mutualistic networks, Nature **550**, 511 (2017).

[12] E. Omodei, M. De Domenico, and A. Arenas, Characterizing interactions in online social networks during exceptional events, Frontiers in Physics **3**, 59 (2015).

[13] C. D. Meyer, *Matrix analysis and applied linear algebra* (SIAM, 2000).

[14] M. G. Everett and S. P. Borgatti, Networks containing negative ties, Social Networks **38**, 111 (2014).

[15] C. H. Hubbell, An input-output approach to clique identification, Sociometry **28**, 377 (1965).

[16] J. Hofbauer and K. Sigmund, *Evolutionary games and population dynamics* (Cambridge University Press, 1998).

[17] S. Kéfi, Ecological networks: from structure to dynamics, in *Theoretical ecology: concepts and applications*, edited by K. S. McCann and G. Gellner (Oxford University Press, 2020) Chap. 10.

[18] M. Novak, J. D. Yeakel, A. E. Noble, D. F. Doak, M. Emmerson, J. A. Estes, U. Jacob, M. T. Tinker, and J. T. Wootton, Characterizing species interactions to understand press perturbations: what is the community matrix?, Annual Review of Ecology, Evolution, and Systematics **47**, 409 (2016).

[19] C. Song and S. Saavedra, Will a small randomly assembled community be feasible and stable?, Ecology **99**, 743 (2018).

[20] J. H. Vandermeer, Interspecific competition: a new approach to the classical theory, Science **188**, 253 (1975).

[21] Y. R. Zelnik, N. Galiana, M. Barbier, M. Loreau, E. Galbraith, and J.-F. Arnoldi, How collectively integrated are ecological communities?, Ecology Letters **27**, e14358 (2024).

[22] P. Bonacich and P. Lloyd, Calculating status with negative relations, Social networks **26**, 331 (2004).

[23] S. F. Sampson, *A novitiate in a period of change: An experimental and case study of social relationships* (Dissertation, Cornell University, 1968).

[24] M. Girvan and M. E. Newman, Community structure in social and biological networks, Proceedings of the national academy of sciences **99**, 7821 (2002).

[25] A.-M. Neutel, J. A. Heesterbeek, and P. C. De Ruiter, Stability in real food webs: weak links in long loops, Science **296**, 1120 (2002).

[26] D. García-Callejas, O. Godoy, L. Buche, M. Hurtado, J. B. Lanuza, A. Allen-Perkins, and I. Bartomeus, Non-random interactions within and across guilds shape the potential to coexist in multi-trophic ecological communities, Ecology Letters **26**, 831 (2023).

[27] G. Gellner and K. S. McCann, Consistent role of weak and strong interactions in high-and low-diversity trophic food webs, Nature communications **7**, 11180 (2016).

[28] R. O. Moura, V. Calleja Solanas, J. A. Langa, and O. Godoy, A general framework for cycles in ecology, bioRxiv 10.1101/2025.01.13.632680 (2025).